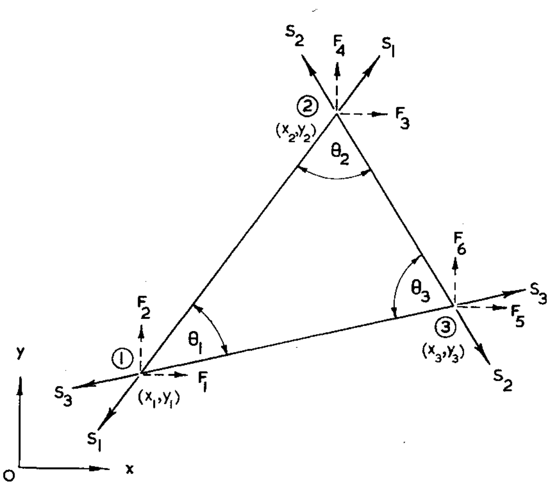

Fig. 1 Independent force systems S = $\{S_1S_2S_3\}$ on a triangular plate element.

sides of triangular elements such that when adjacent sides of two triangular plate elements meet, the corresponding framework elements are represented by two parallel pin-jointed bars. However, in contrast to a real pin-jointed framework, where the flexibility of each unassembled bar element is independent of other elements, the flexibilities of bar elements in the idealized framework are coupled in sets of three bars representing the three sides of a triangular plate element.

The self-equilibrating (redundant) force systems in the idealized structure can be interpreted as sets of forces arising from unit forces acting across fictitious cuts that are selected in such a way that the idealized structure is reduced to a statically determinate one. These systems form simple patterns in a structure made up by triangular plate elements. Some such systems for two-dimensional structures are shown in Fig. 2. The two-element systems consist simply of a pair of S forces belonging to two adjacent triangular elements (Fig. 2a). Systems involving three or four triangular plate elements are shown in Figs. 2b and 2c. Naturally, systems comprising more than four elements also can be constructed.

a) 2-element system

b) 3-element system

c) 4-element system

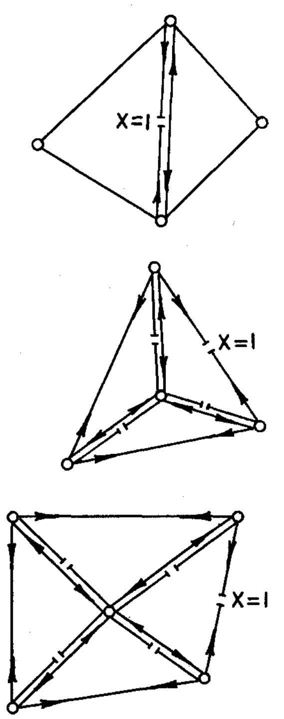

Fig. 2 Self-equilibrating force systems in triangular plate elements (two-dimensional structures).

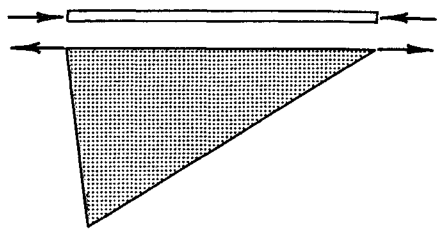

Fig. 3 Self-equilibrating force system consisting of a triangular plate and edge stiffener.

The two-element system can be used even if two adjacent plates are inclined. Furthermore, a system comprising only one triangular plate element and a bar element can be constructed as shown in Fig. 3. Complex three-dimensional structures, however, would require other types of self-equilibrating systems which can be best generated from the equations of equilibrium for the element forces S.

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A Property of Optimum Paths Common to Newtonian and Uniform Force Fields

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THIS paper is concerned with the path of a vehicle which is required to pass through two prescribed points, P_1 and P_2, as shown in Fig. 1. For a given force field, this requirement imposes a relation between the speed and direction of motion at P_1. Three force fields are studied: 1) the Newtonian or inverse square force field, 2) the uniform parallel force field, and 3) the linear central force field. In each field the path is found for which the speed V_1 is a minimum. For the first two fields, which are used most often in preliminary analyses, it is shown that the associated flight path angles are identical. A study of the third field shows that this result is not true for general radially dependent central force fields.

In a Newtonian force field the vehicle path is determined in terms of the initial conditions by the following equations:

$$u'' + u = \mu/r_1^2V_1^2 \cos^2\alpha_1 \tag{1a}$$

and at $\theta = \theta_1 = 0$,

$$u = 1/r_1$$
$$u' = -(1/r_1) \tan\alpha_1 \tag{1b}$$

where $u = 1/r$, a prime indicates differentiation with respect

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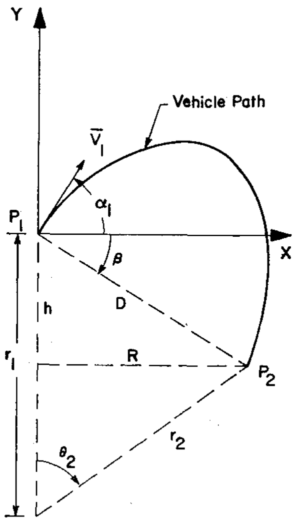

Fig. 1 Geometry and nomenclature.

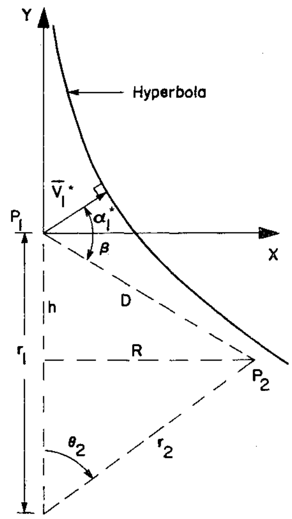

Fig. 2 Optimum α_1* for minimum velocity.

to θ, and μ is the gravitational constant. The general solution for the path is

$$u = \frac{\mu}{r_1{}^2 V_1{}^2 \cos^2\alpha_1} + \left(\frac{1}{r_1} - \frac{\mu}{r_1{}^2 V_1{}^2 \cos^2\alpha_1}\right)\cos\theta - \frac{1}{r_1}\tan\alpha_1 \sin\theta \quad (2)$$

If the path passes through P_2, Eq. (2) indicates that the quantities V_1 and α_1 must be related in the following manner:

$$(r_1 - r_2 \cos\theta_2) V_1{}^2 \cos^2\alpha_1 + r_2 \sin\theta_2 \, V_1{}^2 \sin\alpha_1 \cos\alpha_1 = (r_2\mu/r_1)(1 - \cos\theta_2) \quad (3)$$

This relation is used more readily if it is expressed in terms of the rectangular components of velocity. Using

$$X = V_1 \cos\alpha_1 \qquad Y = V_1 \sin\alpha_1 \quad (4)$$

the relation (3) becomes

$$(r_1 - r_2 \cos\theta_2)X^2 + r_2 \sin\theta_2 XY = (r_2\mu/r_1)(1 - \cos\theta_2) \quad (5)$$

or equivalently

$$hX^2 + RXY = (r_2\mu/r_1)(1 - \cos\theta_2) \quad (6)$$

Thus, in order to pass from P_1 to P_2, the end point of the velocity vector must lie on the hyperbola given by Eq. (5). It easily is established that the asymptotes are the vertical axis and the line between the two points:

$$\frac{Y}{X} = \frac{-(r_1 - r_2 \cos\theta_2)}{r_2 \sin\theta_2} = \frac{-h}{R} \quad (7)$$

as shown in Fig. 2. For $0 < \theta_2 < 180°$, the hyperbola is above $\langle P_1P_2 \rangle$. Since the nearest point of the hyperbola to the center of the X-Y coordinate system is the vertex, the optimal direction bisects the angle between the asymptotes. That is,

$$\alpha_1{}^* = (\pi/4) - (\beta/2) \quad (8)$$

where

$$\tan\beta = \frac{r_1 - r_2 \cos\theta_2}{r_2 \sin\theta_2} = \frac{h}{R} \quad (9)$$

Several points of interest follow easily from the foregoing results. First, whereas the minimum velocity required to reach P_2 from P_1 depends on the distance between P_1 and P_2, the distance to the center (r_1), and the intensity of the field, the associated flight path angle depends on none of these but is given completely by the single geometric parameter β. Thus, all points P_2 on a ray drawn from P_1 have the same flight path angle associated with the minimum speeds required to reach those points. Second, by allowing the force center to recede to infinity while keeping P_1 and P_2 fixed and holding $\mu/r_1{}^2 = g$, the corresponding results may be found for a uniform parallel force field. Since this process does not involve β, it is clear that the optimal flight path angle for this second field is the same as that for the Newtonian field. Third, if P_2 is on the horizon as seen from $P_1(\beta = 0)$, then the optimal flight path angle for a Newtonian field is 45°, the well-known result for parallel force fields. In addition, if P_2 is above or below the horizon as seen from P_1, then the optimal flight path is, respectively, above or below 45°, as is seen from Eq. (8). More specifically, α_1* may be found from

$$\tan 2\alpha_1{}^* = \frac{r_2 \sin\theta_2}{r_1 - r_2 \cos\theta_2} = \frac{R}{h} \quad (10)$$

or, alternatively,

$$\tan\alpha_1{}^* = \frac{D + r_2 \cos\theta_2 - r_1}{r_2 \sin\theta_2} = \frac{D - h}{R} \quad (11)$$

where

$$D = (r_1{}^2 + r_2{}^2 - 2r_1r_2 \cos\theta_2)^{1/2} = (R^2 + h^2)^{1/2} \quad (12)$$

Fourth, the minimum speed required to reach P_2 from P_1 is as follows:

1) In a Newtonian force field,

$$V_1{}^* = \left[\frac{2\mu}{r_1r_2}\left(\frac{D + r_2 \cos\theta_2 - r_1}{1 + \cos\theta_2}\right)\right]^{1/2} \quad (13)$$

2) In a parallel constant force field,

$$V_1{}^* = [g(D - h)]^{1/2} \quad (14)$$

Because of the agreement between the optimal directions in the foregoing two fields, one may be led to suspect that the same result holds for all attractive force fields dependent only on the radius. In support of this idea is the fact that the asymptote established by the line from P_1 to P_2 is common to all such fields. In addition, as the center of force recedes to infinity, such fields should have the uniform parallel force field as a common limit. However, that this is not true for general r-dependent central force fields easily is shown by considering the counterexample provided by the linear field $P = k^2r$. For this field the relation between X and Y is

$$(r_1{}^2 - r_2{}^2 \cos^2\theta_2)X^2 + 2R \, r_2 \cos\theta_2 \, XY - R^2Y^2 = k^2R^2r_1{}^2 \quad (15)$$

which is again a hyperbola and has as asymptotes the lines

$$\frac{Y}{X} = \frac{-(r_1 - r_2 \cos\theta_2)}{r_2 \sin\theta_2} = \frac{-h}{R} \tag{16}$$

and

$$\frac{Y}{X} = \frac{(r_1 + r_2 \cos\theta_2)}{r_2 \sin\theta_2} = \frac{2r_1 - h}{R} \tag{17}$$

The first asymptote is the line $\langle P_1P_2 \rangle$, as found before. However, the vertical axis is not an asymptote in this case. Consequently, the direction for which minimum speed is obtained is different from the common result of the Newtonian and uniform fields.

Coordinate Perturbations from Kepler Orbits

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1. Introduction

IN the astrodynamics literature various perturbation analyses in terms of geometrically obvious coordinates have enjoyed some popularity in connection with guidance work. In particular, one is concerned with the perturbations resulting from position and velocity errors at certain points along the orbit, which sometimes are referred to as "orbit sensitivities." These formulations contrast somewhat with the classical perturbation analyses in terms of the osculating elements or other sets of canonical variables.

At an earlier occasion the author examined the various perturbations of a satellite orbit in terms of a moving coordinate system (Fig. 1) centered at the nominal instantaneous satellite position 0′. The position and velocity of the vehicle in terms of ξ, η, ζ describe its actual motion relative to the "nominal" orbit.† The latter may have been the aim of an imperfect guidance maneuver or actually may have existed prior to whatever physical disturbances brought the discrepancies ξ, η, ζ into being.

A study of satellite orbits according to this approach yielded first-order perturbations of near-circular paths.[1, 2] Ever since that time an interest has persisted in the extension of this formulation to orbits with large eccentricities. The following note records the solutions of the homogenous form of the governing differential equations for any elliptic, near-parabolic, or hyperbolic orbit.

In keeping with the spirit of the earlier work, these solutions were derived from the simple geometric facts that they represent. To be sure, one could extract the same results from the more sophisticated and geometrically more remote formalism of Hamiltonian mechanics.‡ The use of canonical transformations becomes unavoidable in progressing from the present complementary solutions to particular ones if the algebra is to remain tolerable.

The complementary solutions themselves, as given in this note, may be used to study the divergence of transfer orbits from the earth to the moon or to another planet due to guidance errors. They will also describe the spread in descent trajectories to the lunar surface or re-entry in the outer fringes of the earth's atmosphere due to anomalies in vehicle position or velocity at the start of such maneuvers. In many applications an accurate description of perturbative displacements cannot be given without including the sustained effects of drag, a nonspherical central body, and extraneous gravitation. The necessary extensions of the present formulation are currently under study; they can be regarded as a logical refinement of the popular patched-conic approach to multiphase orbits.

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† For a later treatment of a similar approach, see Ref. 3.

‡ In fact, some development in this direction from Brouwer's perturbation method has been reported to the author by D. Brouwer and H. R. Westerman.

2. Elliptic Orbits of Arbitrary Eccentricity

The equations of motion in terms of ξ, η, ζ have been derived for nonvanishing eccentricity in Ref. 1 and will be merely restated here:

$$\xi'' - 2\eta' - \xi - 2[\xi + e(\xi' - \eta) \sin f]/(1 + e \cos f) = 0 \tag{1}$$

$$\eta'' + 2\xi' - \eta - [-\eta + 2e(\eta' + \xi) \sin f]/(1 + e \cos f) = 0 \tag{2}$$

$$\zeta'' + [\zeta - 2\zeta' e \sin f]/(1 + e \cos f) = 0 \tag{3}$$

where the primes denote derivatives with respect to the true anomaly f of the nominal orbit.

At first sight these equations appear sufficiently awkward to be discouraging; but one notes that they apply equally well to elliptic and hyperbolic orbits, and, with $e = 1$, they are valid for parabolic orbits. In the derivation of these equations the usual linearizations were employed, and hence their solutions must be regarded as a first-order representation of the perturbations in a geometric sense. It stands to reason that such solutions should be obtainable from the first derivatives of the position in a Kepler orbit with respect to the orbit parameters (linear sensitivities). This is the rationale to be followed below.

Remembering the basic formulas of an elliptic orbit, one has

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \qquad x = r \cos f \qquad y = r \sin f$$

or, in terms of the eccentric anomaly,

$$r = a(1 - e \cos E) \qquad x = a(\cos E - e)$$
$$y = a(1 - e^2)^{1/2} \sin E$$

The x axis points from the focus through pericenter, and the y axis coincides with $f = \pi/2$. From Kepler's equation, one obtains

$$\begin{aligned} \frac{\partial E}{\partial a} &= -\frac{3}{2a} \frac{E - e \sin E}{1 - e \cos E} \\ \frac{\partial E}{\partial e} &= \frac{\sin E}{1 - e \cos E} \\ \frac{\partial E}{\partial \tau} &= \frac{(k/a^3)^{1/2}}{1 - e \cos E} \end{aligned} \tag{4}$$

where k is the gravitational constant multiplied by the central mass. Thus $\partial x/\partial a, \ldots, \partial y/\partial \tau$ follow, and if one denotes

$$\delta x = \frac{\partial x}{\partial a} \delta a + \frac{\partial x}{\partial e} \delta e + \frac{\partial x}{\partial \tau} \delta \tau$$

$$\delta y = \frac{\partial y}{\partial a} \delta a + \frac{\partial y}{\partial e} \delta e + \frac{\partial y}{\partial \tau} \delta \tau$$

then

$$\begin{aligned} \xi &= \delta x \cos f + \delta y \sin f \\ \eta &= -\delta x \sin f + \delta y \cos f + r\delta\omega \end{aligned} \tag{5}$$

The δ's represent small changes, and the last term in the expression for η represents a rigid-body rotation of the entire